# Minimal distance to a cubic function

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# Introduction

When you want to develop a selfdriving car, you have to plan which path it should take. A reasonable choice for the representation of paths are cubic splines. You also have to be able to calculate how to steer to get or to remain on a path. A way to do this is applying the PID algorithm. This algorithm needs to know the signed current error. So you need to be able to get the minimal distance of a point to a cubic spline combined with the direction (left or right). As you need to get the signed error (and one steering direction might be prefered), it is not only necessary to get the minimal absolute distance, but might also help to get all points on the spline with minimal distance.

In this paper I want to discuss how to find all points on a cubic function with minimal distance to a given point. As other representations of paths might be easier to understand and to implement, I will also cover the problem of finding the minimal distance of a point to a polynomial of degree 0, 1 and 2.

While I analyzed this problem, I've got interested in variations of the underlying PID-related problem. So I will try to give robust and easy-to-implement algorithms to calculated the distance of a point to a (piecewise or global) defined polynomial function of degree $\leq 3$.

# Contents

# 1 Description of the Problem

Let $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ be a polynomial function and $P \in \mathbb{R}^2$ be a point. Let $d_{P,f} : \mathbb{R} \to \mathbb{R}_0^+$ be the Euklidean distance of a point $P$ to a point $(x, f(x))$ on the graph of $f$:

$$d_{P,f}(x) := \sqrt{(x_P - x)^2 + (y_P - f(x))^2}$$

Now there is finite set $M = \{ x_1, \dots, x_n \} \subseteq D$ of minima for given $f$ and $P$:

$$M = \left\{ x \in D \;\middle|\; d_{P,f}(x) = \min_{\overline{x} \in D} d_{P,f}(\overline{x}) \right\}$$

But minimizing $d_{P,f}$ is the same as minimizing $d_{P,f}^2 = x_p^2 - 2x_p x + x^2 + y_p^2 - 2y_p f(x) + f(x)^2$.

In order to solve the minimal distance problem, Fermat's theorem about stationary points will be tremendously usefull:

**Theorem 1 (Fermat's theorem about stationary points)**
Let $x_0$ be a local extremum of a differentiable function $f : \mathbb{R} \to \mathbb{R}$.

Then: $f'(x_0) = 0$.

Let $S_n$ be the function that returns the set of solutions for a polynomial $f$ of degree $n$ and a point $P$:

$$S_n : \{ \text{Polynomials of degree } n \text{ defined on } \mathbb{R} \} \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$$

$$S_n(f, P) := \arg\min_{x \in \mathbb{R}} d_{P,f}(x) = M$$

If possible, I will explicitly give this function.

# 2 Constant functions

## 2.1 Defined on $\mathbb{R}$

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) := c$ with $c \in \mathbb{R}$ be a constant function. The situation can be seen in Figure 2.1.

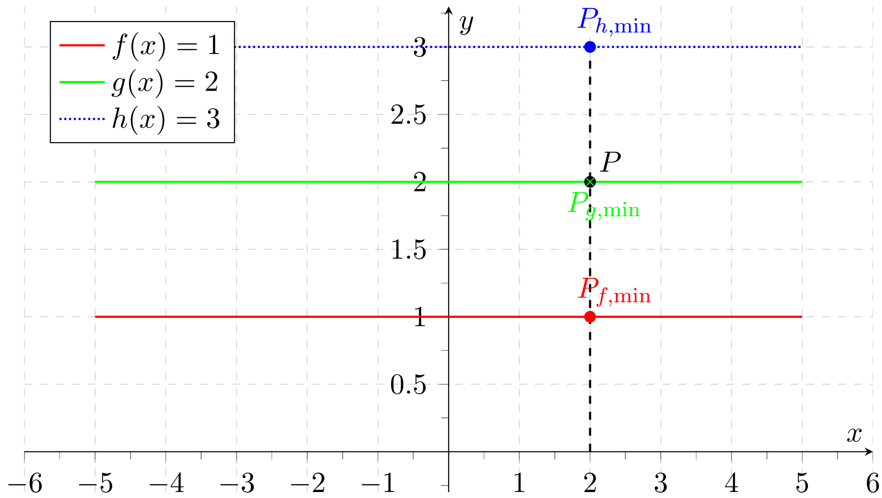


Figure 2.1: Three constant functions and their points with minimal distance

$$
\begin{aligned}
d_{P,f}(x) &= \sqrt{(x_P - x)^2 + (y_P - f(x))^2} && (2.1)\\
&= \sqrt{(x_P^2 - 2x_P x + x^2) + (y_P^2 - 2y_P c + c^2)} && (2.2)\\
&= \sqrt{x^2 - 2x_P x + (x_P^2 + y_P^2 - 2y_P c + c^2)} && (2.3)\\
\xRightarrow{\text{Theorem 1}} 0 &\overset{!}{=} (d_{P,f}(x)^2)' && (2.4)\\
&= 2x - 2x_P && (2.5)\\
\Leftrightarrow x &\overset{!}{=} x_P && (2.6)
\end{aligned}
$$

Then $(x_P, f(x_P))$ has minimal distance to $P$. Every other point has higher distance. See Figure 2.1 to see that intuition yields to the same results.

This result means:

$$S_0(f, P) = \{ x_P \} \text{ with } P = (x_P, y_P)$$

## 2.2 Defined on a closed interval $[a, b] \subseteq \mathbb{R}$

Let $f : [a, b] \to \mathbb{R}$, $f(x) := c$ with $a, b, c \in \mathbb{R}$ and $a \leq b$ be a constant function.

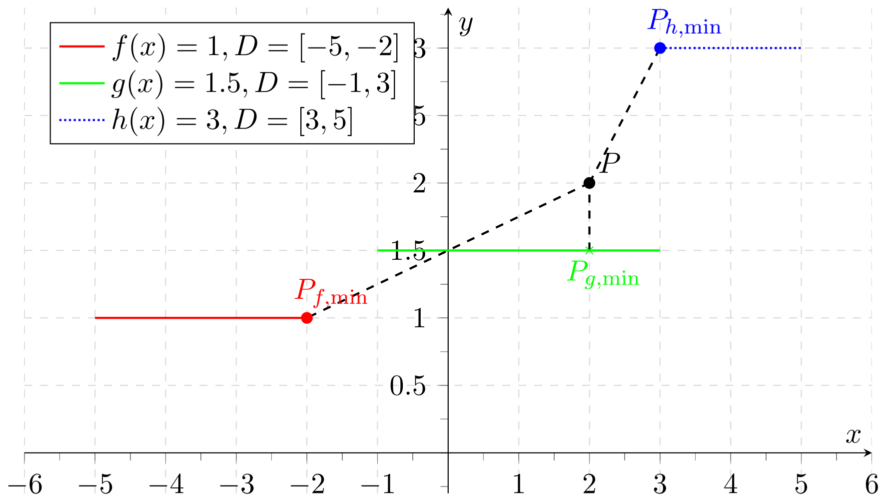


Figure 2.2: Three constant functions and their points with minimal distance

The point with minimum distance can be found by:

$$\arg\min_{x \in \mathbb{R}} d_{P,f}(x) = \begin{cases} S_0(f, P) & \text{if } S_0(f, P) \cap [a, b] \neq \emptyset \\ \{ a \} & \text{if } S_0(f, P) \ni x_P < a \\ \{ b \} & \text{if } S_0(f, P) \ni x_P > b \end{cases}$$

Because:

$$\arg\min_{x \in [a,b]} d_{P,f}(x) = \arg\min_{x \in [a,b]} d_{P,f}(x)^2 \tag{2.7}$$

$$= \arg\min_{x \in [a,b]} x^2 - 2x_P x + (x_P^2 + y_P^2 - 2y_P c + c^2) \tag{2.8}$$

$$= \arg\min_{x \in [a,b]} x^2 - 2x_P x + x_P^2 \tag{2.9}$$

$$= \arg\min_{x \in [a,b]} (x - x_P)^2 \tag{2.10}$$

# 3 Linear function

## 3.1 Defined on $\mathbb{R}$

Let $f(x) = m \cdot x + t$ with $m \in \mathbb{R} \setminus \{ 0 \}$ and $t \in \mathbb{R}$ be a linear function.

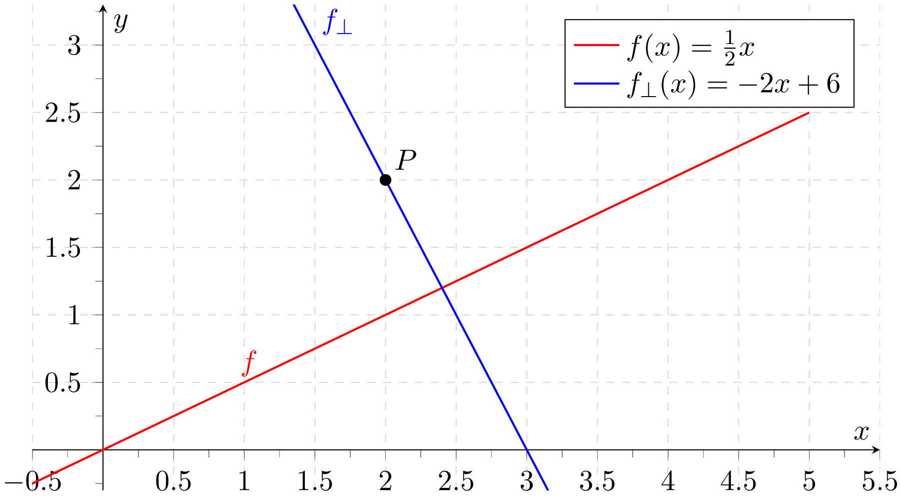


Figure 3.1: The shortest distance of $P$ to $f$ can be calculated by using the perpendicular

Now you can drop a perpendicular $f_\perp$ through $P$ on $f(x)$. The slope of $f_\perp$ is $-\frac{1}{m}$ and $t_\perp$ can be calculated:

$$f_\perp(x) = -\frac{1}{m} \cdot x + t_\perp \tag{3.1}$$

$$\Rightarrow y_P = -\frac{1}{m} \cdot x_P + t_\perp \tag{3.2}$$

$$\Leftrightarrow t_\perp = y_P + \frac{1}{m} \cdot x_P \tag{3.3}$$

The point $(x, f(x))$ where the perpendicular $f_\perp$ crosses $f$ is calculated this way:

$$f(x) = f_\perp(x) \tag{3.4}$$

$$\Leftrightarrow m \cdot x + t = -\frac{1}{m} \cdot x + \left( y_P + \frac{1}{m} \cdot x_P \right) \tag{3.5}$$

$$\Leftrightarrow \left( m + \frac{1}{m} \right) \cdot x = y_P + \frac{1}{m} \cdot x_P - t \tag{3.6}$$

$$\Leftrightarrow x = \frac{m}{m^2 + 1} \left( y_P + \frac{1}{m} \cdot x_P - t \right) \tag{3.7}$$

There is only one point with minimal distance. I'll call the result from line 3.7 "solution of the linear problem" and the function that gives this solution $S_1(f, P)$.

See Figure 3.1 to get intuition about the geometry used.

## 3.2 Defined on a closed interval $[a, b] \subseteq \mathbb{R}$

Let $f : [a, b] \to \mathbb{R}$, $f(x) := m \cdot x + t$ with $a, b, m, t \in \mathbb{R}$ and $a \leq b$, $m \neq 0$ be a linear function.

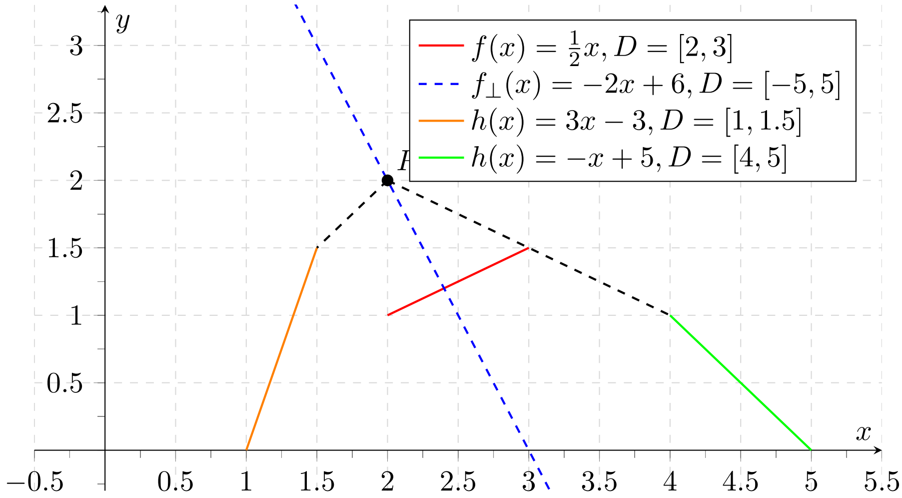


Figure 3.2: Different situations when you have linear functions which are defined on a closed intervall

The point with minimum distance can be found by:

$$\arg\min_{x \in [a,b]} d_{P,f}(x) = \begin{cases} S_1(f, P) & \text{if } S_1(f, P) \cap [a, b] \neq \emptyset \\ \{\, a \,\} & \text{if } S_1(f, P) \ni x < a \\ \{\, b \,\} & \text{if } S_1(f, P) \ni x > b \end{cases}$$

argument? proof?

# 4 Quadratic functions

## 4.1 Defined on $\mathbb{R}$

Let $f(x) = a \cdot x^2 + b \cdot x + c$ with $a \in \mathbb{R} \setminus \{ 0 \}$ and $b, c \in \mathbb{R}$ be a quadratic function.

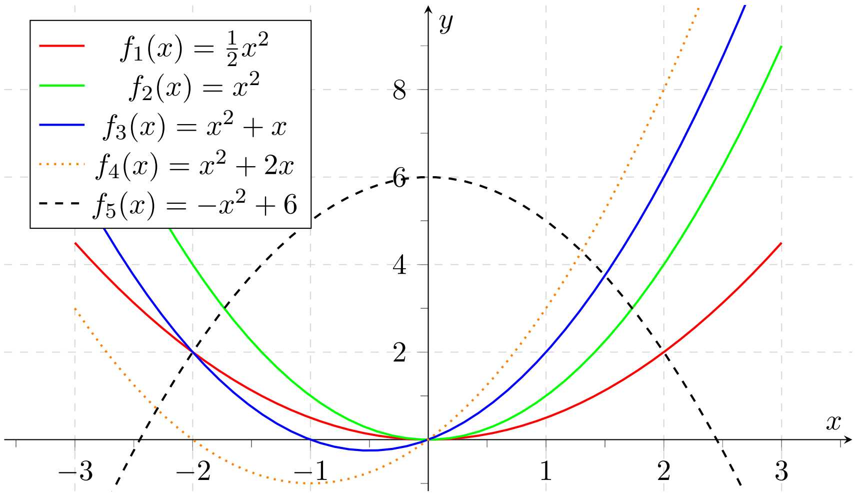


Figure 4.1: Quadratic functions

### 4.1.1 Calculate points with minimal distance

In this case, $d_{P,f}^2$ is polynomial of degree 4. We use Theorem **??**:

$$
\begin{aligned}
0 &\stackrel{!}{=} (d_{P,f}^2)' && (4.1)\\
&= -2x_p + 2x - 2y_p f'(x) + \big(f(x)^2\big)' && (4.2)\\
&= -2x_p + 2x - 2y_p f'(x) + 2f(x) \cdot f'(x) \qquad \text{(chain rule)} && (4.3)\\
\Leftrightarrow 0 &\stackrel{!}{=} -x_p + x - y_p f'(x) + f(x) \cdot f'(x) \qquad \text{(divide by 2)} && (4.4)\\
&= -x_p + x - y_p(2ax + b) + (ax^2 + bx + c)(2ax + b) && (4.5)\\
&= -x_p + x - y_p \cdot 2ax - y_p b + (2a^2x^3 + 2abx^2 + 2acx + abx^2 + b^2x + bc) && (4.6)\\
&= -x_p + x - 2y_p ax - y_p b + (2a^2x^3 + 3abx^2 + 2acx + b^2x + bc) && (4.7)\\
&= 2a^2x^3 + 3abx^2 + (1 - 2y_p a + 2ac + b^2)x + (bc - by_p - x_p) && (4.8)
\end{aligned}
$$

This is an algebraic equation of degree 3. There can be up to 3 solutions in such an equation. Those solutions can be found with a closed formula. But not every solution of the equation given by Theorem **??** has to be a solution to the given problem.

**Example 1**

Let $a = 1, b = 0, c = 1, x_p = 0, y_p = 1$. So $f(x) = x^2 + 1$ and $P(0, 1)$.

$$0 \stackrel{!}{=} 4x^3 - 2x \tag{4.9}$$
$$= 2x(2x^2 - 1) \tag{4.10}$$
$$\Rightarrow x_1 = 0 \quad x_{2,3} = \pm\frac{1}{\sqrt{2}} \tag{4.11}$$

As you can easily verify, only $x_1$ is a minimum of $d_{P,f}$.

### 4.1.2 Number of points with minimal distance

**Theorem 2**
A point $P$ has either one or two points on the graph of a quadratic function $f$ that are closest to $P$.

**Proof:** In the following, I will do some transformations with $f = f_0$ and $P = P_0$ .

Moving $f_0$ and $P_0$ simultaneously in $x$ or $y$ direction does not change the minimum distance. Furthermore, we can find the points with minimum distance on the moved situation and calculate the minimum points in the original situation.

First of all, we move $f_0$ and $P_0$ by $\frac{b}{2a}$ in $x$ direction, so

$$f_1(x) = ax^2 - \frac{b^2}{4a} + c \quad \text{and} \quad P_1 = \left(x_p + \frac{b}{2a},\ y_p\right)$$

Because:[1]

$$f(x - {}^{b}\!/_{2a}) = a(x - {}^{b}\!/_{2a})^2 + b(x - {}^{b}\!/_{2a}) + c \tag{4.12}$$
$$= a(x^2 - {}^{b}\!/_{a}x + {}^{b^2}\!/_{4a^2}) + bx - {}^{b^2}\!/_{2a} + c \tag{4.13}$$
$$= ax^2 - bx + {}^{b^2}\!/_{4a} + bx - {}^{b^2}\!/_{2a} + c \tag{4.14}$$
$$= ax^2 - {}^{b^2}\!/_{4a} + c \tag{4.15}$$

Then move $f_1$ and $P_1$ by $\frac{b^2}{4a} - c$ in $y$ direction. You get:

$$f_2(x) = ax^2 \quad \text{and} \quad P_2 = \left(\underbrace{x_P + \frac{b}{2a}}_{=:z},\ \underbrace{y_P + \frac{b^2}{4a} - c}_{=:w}\right)$$

As $f_2(x) = ax^2$ is symmetric to the $y$ axis, only points $P = (0, w)$ could possilby have three minima.

[1] The idea why you subtract $\frac{b}{2a}$ within $f$ is that when you subtract something from $x$ before applying $f$ it takes more time ($x$ needs to be bigger) to get to the same situation. So to move the whole graph by 1 to the left whe have to add +1.

Then compute:

$$
\begin{aligned}
d_{P,f_2}(x) &= \sqrt{(x-0)^2 + (f_2(x) - w)^2} && (4.16)\\
&= \sqrt{x^2 + (ax^2 - w)^2} && (4.17)\\
&= \sqrt{x^2 + a^2x^4 - 2awx^2 + w^2} && (4.18)\\
&= \sqrt{a^2x^4 + (1-2aw)x^2 + w^2} && (4.19)\\
&= \sqrt{\left(a^2x^2 + \frac{1-2aw}{2}\right)^2 + w^2 - (1-2aw)^2} && (4.20)\\
&= \sqrt{\left(a^2x^2 + 1/2 - aw\right)^2 + \left(w^2 - (1-2aw)^2\right)} && (4.21)
\end{aligned}
$$

The term

$$a^2x^2 + (1/2 - aw)$$

should get as close to 0 as possilbe when we want to minimize $d_{P,f_2}$. For $w \leq 1/2a$ you only have $x = 0$ as a minimum. For all other points $P = (0, w)$, there are exactly two minima $x_{1,2} = \pm\sqrt{aw - 1/2}$. ■

### 4.1.3 Solution formula

We start with the graph that was moved so that $f_2 = ax^2$.

**Case 1:** $P$ is on the symmetry axis, hence $x_P = -\frac{b}{2a}$.

In this case, we have already found the solution. If $y_P + \frac{b^2}{4a} - c > \frac{1}{2a}$, then there are two solutions:

$$x_{1,2} = \pm\sqrt{aw - 1/2}$$

Otherwise, there is only one solution $x_1 = 0$.

**Case 2:** $P = (z, w)$ is not on the symmetry axis, so $z \neq 0$. Then you compute:

$$
\begin{aligned}
d_{P,f_2}(x) &= \sqrt{(x-z)^2 + (f(x) - w)^2} && (4.22)\\
&= \sqrt{(x^2 - 2zx + z^2) + ((ax^2)^2 - 2awx^2 + w^2)} && (4.23)\\
&= \sqrt{a^2x^4 + (1-2aw)x^2 + (-2z)x + z^2 + w^2} && (4.24)\\
0 &\overset{!}{=} \left(\left(d_{P,f_2}(x)\right)^2\right)' && (4.25)\\
&= 4a^2x^3 + 2(1-2aw)x + (-2z) && (4.26)\\
&= 2\left(2a^2x^2 + (1-2aw)\right)x - 2z && (4.27)\\
\Leftrightarrow 0 &\overset{!}{=} (2a^2x^2 + (1-2aw))x - z && (4.28)\\
&= 2a^2x^3 + (1-2aw)x - z && (4.29)\\
\Leftrightarrow 0 &\overset{!}{=} x^3 + \underbrace{\frac{1-2aw}{2a^2}}_{=:\alpha} x + \underbrace{\frac{-z}{2a^2}}_{=:\beta} && (4.30)\\
&= x^3 + \alpha x + \beta && (4.31)
\end{aligned}
$$

Let $t$ be defined as

$$t := \sqrt[3]{\sqrt{3 \cdot (4\alpha^3 + 27\beta^2)} - 9\beta}$$

**Case 2.1:** $4\alpha^3 + 27\beta^2 \geq 0$: The solution of Equation 4.31 is

$$x = \frac{t}{\sqrt[3]{18}} - \frac{\sqrt[3]{\frac{2}{3}}\alpha}{t}$$

When you insert this in Equation 4.31 you get:[2]

$$\begin{aligned}
0 &\stackrel{!}{=} \left(\frac{t}{\sqrt[3]{18}} - \frac{\sqrt[3]{\frac{2}{3}}\alpha}{t}\right)^3 + \alpha\left(\frac{t}{\sqrt[3]{18}} - \frac{\sqrt[3]{\frac{2}{3}}\alpha}{t}\right) + \beta && (4.32)\\
&= (\frac{t}{\sqrt[3]{18}})^3 - 3(\frac{t}{\sqrt[3]{18}})^2\frac{\sqrt[3]{\frac{2}{3}}\alpha}{t} + 3(\frac{t}{\sqrt[3]{18}})(\frac{\sqrt[3]{\frac{2}{3}}\alpha}{t})^2 - (\frac{\sqrt[3]{\frac{2}{3}}\alpha}{t})^3 + \alpha\left(\frac{t}{\sqrt[3]{18}} - \frac{\sqrt[3]{\frac{2}{3}}\alpha}{t}\right) + \beta && (4.33)\\
&= \frac{t^3}{18} - \frac{3t^2}{\sqrt[3]{18^2}}\frac{\sqrt[3]{\frac{2}{3}}\alpha}{t} + \frac{3t}{\sqrt[3]{18}}\frac{\sqrt[3]{\frac{4}{9}}\alpha^2}{t^2} - \frac{\frac{2}{3}\alpha^3}{t^3} + \alpha\left(\frac{t}{\sqrt[3]{18}} - \frac{\sqrt[3]{\frac{2}{3}}\alpha}{t}\right) + \beta && (4.34)\\
&= \frac{t^3}{18} - \frac{\sqrt[3]{18}t\alpha}{\sqrt[3]{18^2}} + \frac{\sqrt[3]{12}\alpha^2}{\sqrt[3]{18}t} - \frac{\frac{2}{3}\alpha^3}{t^3} + \alpha\left(\frac{t}{\sqrt[3]{18}} - \frac{\sqrt[3]{\frac{2}{3}}\alpha}{t}\right) + \beta && (4.35)\\
&= \frac{t^3}{18} - \frac{t\alpha}{\sqrt[3]{18}} + \frac{\sqrt[3]{2}\alpha^2}{\sqrt[3]{3}t} - \frac{\frac{2}{3}\alpha^3}{t^3} + \alpha\left(\frac{t}{\sqrt[3]{18}} - \frac{\sqrt[3]{\frac{2}{3}}\alpha}{t}\right) + \beta && (4.36)\\
&= \frac{t^3}{18} - \frac{t\alpha}{\sqrt[3]{18}} - \frac{\frac{2}{3}\alpha^3}{t^3} + \frac{\alpha t}{\sqrt[3]{18}} + \beta && (4.37)\\
&= \frac{t^3}{18} - \frac{\frac{2}{3}\alpha^3}{t^3} + \beta && (4.38)\\
&= \frac{t^6 - 12\alpha^3 + \beta 18t^3}{18t^3} && (4.39)
\end{aligned}$$

Now only go on calculating with the numerator. Start with resubstituting $t$:

$$\begin{aligned}
0 &= (\sqrt{3 \cdot (4\alpha^3 + 27\beta^2)} - 9\beta)^2 - 12\alpha^3 + \beta 18(\sqrt{3 \cdot (4\alpha^3 + 27\beta^2)} - 9\beta) && (4.40)\\
&= (\sqrt{3 \cdot (4\alpha^3 + 27\beta^2)})^2 + (9\beta)^2 - 12\alpha^3 - 18 \cdot 9\beta^2 && (4.41)\\
&= 3 \cdot (4\alpha^3 + 27\beta^2) - 81\beta^2 - 12\alpha^3 && (4.42)\\
&= (4\alpha^3 + 27\beta^2) - 27\beta^2 - 4\alpha^3 && (4.43)\\
&= 0 && (4.44)
\end{aligned}$$

**Case 2.2:** TODO

$$x = \frac{(1 + i\sqrt{3})a}{\sqrt[3]{12} \cdot t} - \frac{(1 - i\sqrt{3})t}{2\sqrt[3]{18}}$$

**Case 2.3:** TODO

$$x = \frac{(1 - i\sqrt{3})a}{\sqrt[3]{12} \cdot t} - \frac{(1 + i\sqrt{3})t}{2\sqrt[3]{18}}$$

[2] Remember: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

So the solution is given by

NO! Currently, there are erros in the solution. Check $f(x) = x^2$ and $P = (-2, 4)$. Solution should be $x_1 = -2$, but it isn't!

$$
\begin{aligned}
x_S &:= -\frac{b}{2a} \qquad \text{(the symmetry axis)}\\
w &:= y_P + \frac{b^2}{4a} - c \quad \text{and} \quad z := x_P + \frac{b}{2a}\\
\alpha &:= \frac{1-2aw}{2a^2} \quad \text{and} \quad \beta := \frac{-z}{2a^2}\\
t &:= \sqrt[3]{\sqrt{3\cdot(4\alpha^3+27\beta^2)}-9\beta}
\end{aligned}
$$

$$
\arg\min_{x\in\mathbb{R}} d_{P,f}(x) = \begin{cases}
x_1 = +\sqrt{a(y_p + \frac{b^2}{4a} - c) - \frac{1}{2}} + x_S \text{ and} & \text{if } x_P = x_S \text{ and } y_p + \frac{b^2}{4a} - c > \frac{1}{2a}\\
x_2 = -\sqrt{a(y_p + \frac{b^2}{4a} - c) - \frac{1}{2}} + x_S & \\
x_1 = x_S & \text{if } x_P = x_S \text{ and } y_p + \frac{b^2}{4a} - c \leq \frac{1}{2a}\\
x_1 = \frac{t}{\sqrt[3]{18}} - \frac{\sqrt[3]{\frac{2}{3}}\alpha}{t} & \text{if } x_P \neq x_S
\end{cases}
$$

I call this function $S_2 : \{\text{ Quadratic functions }\} \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$.

## 4.2 Defined on a closed interval $[a, b] \subseteq \mathbb{R}$

Now the problem isn't as simple as with constant and linear functions.

If one of the minima in $S_2(P, f)$ is in $[a, b]$, this will be the shortest distance as there are no shorter distances.

The following IS WRONG! Can I include it to help the reader understand the problem?

If the function (defined on $\mathbb{R}$) has only one shortest distance point $x$ for the given $P$, it's also easy: The point in $[a, b]$ that is closest to $x$ will have the sortest distance.

$$\arg\min_{x \in \mathbb{R}} d_{P,f}(x) = \begin{cases} S_2(f, P) \cap [a, b] & \text{if } S_2(f, P) \cap [a, b] \neq \emptyset \\ \{ a \} & \text{if } |S_2(f, P)| = 1 \text{ and } S_2(f, P) \ni x < a \\ \{ b \} & \text{if } |S_2(f, P)| = 1 \text{ and } S_2(f, P) \ni x > b \\ todo & \text{if } |S_2(f, P)| = 2 \text{ and } S_2(f, P) \cap [a, b] = \emptyset \end{cases}$$

# 5 Cubic functions

## 5.1 Defined on $\mathbb{R}$

Let $f(x) = a \cdot x^3 + b \cdot x^2 + c \cdot x + d$ be a cubic function with $a \in \mathbb{R} \setminus \{ 0 \}$ and $b, c, d \in \mathbb{R}$ be a function.

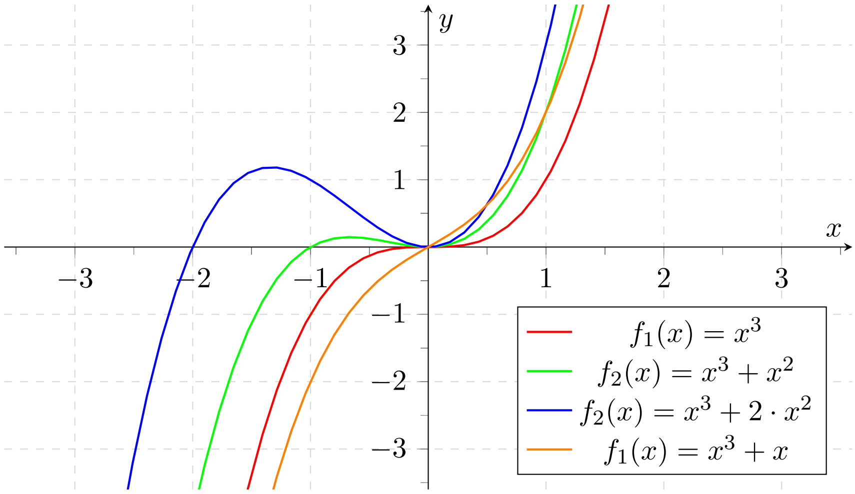


Figure 5.1: Cubic functions

### 5.1.1 Calculate points with minimal distance

**Theorem 3**
There cannot be an algebraic solution to the problem of finding a closest point $(x, f(x))$ to a given point $P$ when $f$ is a polynomial function of degree 3 or higher.

**Proof:** Suppose you could solve the closest point problem for arbitrary cubic functions $f = ax^3 + bx^2 + cx + d$ and arbitrary points $P = (x_P, y_P)$.

Then you could solve the following problem for $x$:

$$\begin{aligned}
0 &\stackrel{!}{=} \left((d_{P,f}(x))^2\right)' \qquad = -2x_p + 2x - 2y_p(f(x))' + (f(x)^2)' && (5.1)\\
&= 2f(x) \cdot f'(x) - 2y_p f'(x) + 2x - 2x_p && (5.2)\\
&= f(x) \cdot f'(x) - y_p f'(x) + x - x_p && (5.3)\\
&= \underbrace{f'(x) \cdot (f(x) - y_p)}_{\text{Polynomial of degree 5}} + x - x_p && (5.4)
\end{aligned}$$

General algebraic equations of degree 5 don't have a solution formula.[1] Although here seems

[1] TODO: Quelle

to be more structure, the resulting algebraic equation can be almost any polynomial of degree 5:[2]

$$
\begin{aligned}
0 &\stackrel{!}{=} f'(x) \cdot (f(x) - y_p) + (x - x_p) && (5.5)\\
&= \underbrace{3a^2}_{=\tilde{a}} x^5 + \underbrace{5ab}_{\tilde{b}} x^4 + \underbrace{2(2ac + b^2)}_{=:\tilde{c}} x^3 \quad + \underbrace{3(ad + bc - ay_p)}_{\tilde{d}} x^2 && (5.6)\\
&\quad + \underbrace{(2bd + c^2 + 1 - 2by_p)}_{=:\tilde{e}} x + \underbrace{cd - cy_p - x_p}_{=:\tilde{f}} && (5.7)\\
0 &\stackrel{!}{=} \tilde{a}x^5 + \tilde{b}x^4 + \tilde{c}x^3 + \tilde{d}x^2 + \tilde{e}x + \tilde{f} && (5.8)
\end{aligned}
$$

1. For any coefficient $\tilde{a} \in \mathbb{R}_{>0}$ of $x^5$ we can choose $a$ such that we get $\tilde{a}$.
2. For any coefficient $\tilde{b} \in \mathbb{R} \setminus \{ 0 \}$ of $x^4$ we can choose $b$ such that we get $\tilde{b}$.
3. With $c$, we can get any value of $\tilde{c} \in \mathbb{R}$.
4. With $d$, we can get any value of $\tilde{d} \in \mathbb{R}$.
5. With $y_p$, we can get any value of $\tilde{e} \in \mathbb{R}$.
6. With $x_p$, we can get any value of $\tilde{f} \in \mathbb{R}$.

The first restriction guarantees that we have a polynomial of degree 5. The second one is necessary, to get a high range of $\tilde{e}$.

This means, that there is no solution formula for the problem of finding the closest points on a cubic function to a given point, because if there was one, you could use this formula for finding roots of polynomials of degree 5. ■

### 5.1.2 Another approach

Currently, this is only an idea. It might be usefull to move the cubic function $f$ such that $f$ is point symmetric to the origin. But I'm not sure how to make use of this symmetry.

Just like we moved the function $f$ and the point to get in a nicer situation, we can apply this approach for cubic functions.

First, we move $f_0$ by $\frac{b}{3a}$ to the right, so

$$f_1(x) = ax^3 + \frac{b^2(c-1)}{3a}x + \frac{2b^3}{27a^2} - \frac{bc}{3a} + d \quad \text{and} \quad P_1 = (x_P + \frac{b}{3a}, y_P)$$

because

$$
\begin{aligned}
f_1(x) &= a\left(x - \frac{b}{3a}\right)^3 + b\left(x - \frac{b}{3a}\right)^2 + c\left(x - \frac{b}{3a}\right) + d && (5.9)\\
&= a\left(x^3 - 3\frac{b}{3a}x^2 + 3(\frac{b}{3a})^2 x - \frac{b^3}{27a^3}\right) + b\left(x^2 - \frac{2b}{3a}x + \frac{b^2}{9a^2}\right) + cx - \frac{bc}{3a} + d && (5.10)
\end{aligned}
$$

[2]Thanks to Peter Košinár on math.stackexchange.com for this one

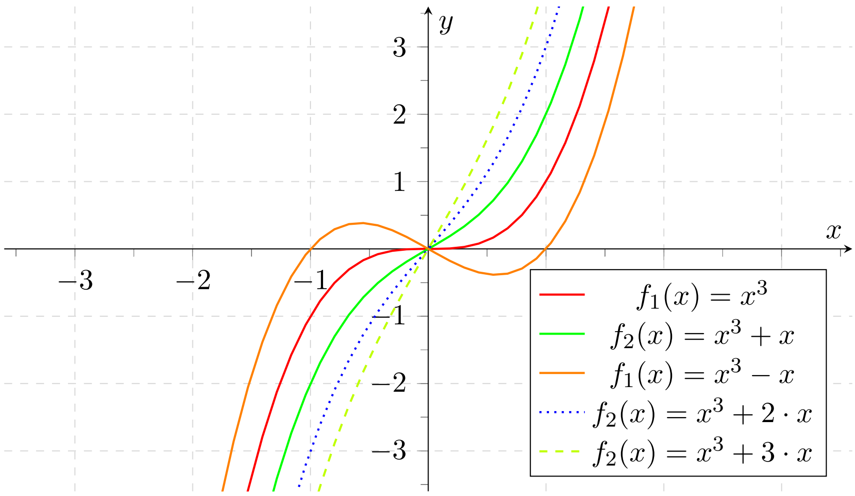


Figure 5.2: Cubic functions with $b = d = 0$

$$= ax^3 - bx^2 + \frac{b^2}{3a}x - \frac{b^3}{27a^2} \tag{5.11}$$

$$+ bx^2 - \frac{2b^2}{3a}x + \frac{b^3}{9a^2} \tag{5.12}$$

$$+ cx - \frac{bc}{3a} + d \tag{5.13}$$

$$= ax^3 + \frac{b^2}{3a}(1 - 2 + c)\,x + \frac{b^3}{9a^2}\left(1 - \frac{1}{3}\right) - \frac{bc}{3a} + d \tag{5.14}$$

### 5.1.3 Number of points with minimal distance

As this leads to a polynomial of degree 5 of which we have to find roots, there cannot be more than 5 solutions.

Can there be 3, 4 or even 5 solutions? Examples!
After looking at function graphs of cubic functions, I'm pretty sure that there cannot be 4 or 5 solutions, no matter how you chose the cubic function $f$ and $P$.
I'm also pretty sure that there is no polynomial (no matter what degree) that has more than 3 solutions.

### 5.1.4 Interpolation and approximation

#### Quadratic spline interpolation

You could interpolate the cubic function by a quadratic spline.

#### Bisection method

TODO

### Newtons method

One way to find roots of functions is Newtons method. It gives an iterative computation procedure that can converge quadratically if some conditions are met:

**Theorem 4 (local quadratic convergence of Newton's method)**
Let $D \subseteq \mathbb{R}^n$ be open and $f : D \to \mathbb{R}^n \in C^2(\mathbb{R})$. Let $x^* \in D$ with $f(x^*) = 0$ and the Jaccobi-Matrix $f'(x^*)$ should not be invertable when evaluated at the root.

Then there is a sphere

$$K := K_\rho(x^*) = \{ x \in \mathbb{R}^n \mid \|x - x^*\|_\infty \leq \rho \} \subseteq D$$

such that $x^*$ is the only root of $f$ in $K$. Furthermore, the elements of the sequence

$$x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$$

are for every starting value $x_0 \in K$ again in $K$ and

$$\lim_{n \to \infty} x_k = x^*$$

Also, there is a constant $C > 0$ such that

$$\|x^* - x_{n+1}\| = C\|x^* - x_n\|^2 \text{ for } n \in \mathbb{N}_0\|$$

The approach is extraordinary simple. You choose a starting value $x_0$ and compute

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

As soon as the values don't change much, you are close to a root. The problem of this approach is choosing a starting value that is close enough to the root. So we have to have a "good" initial guess.

### Quadratic minimization

TODO

## 5.2 Defined on a closed interval $[a, b] \subseteq \mathbb{R}$